1. (a) The perimeter gives $2x + 2y = 16$. Dividing both sides by two, $x + y = 8$.
   (b) Equating expressions for area, $xy = 15$.
   (c) Rearranging the area formula to $y = \frac{15}{x}$, we substitute for y:
$$\begin{aligned} & x + \tfrac{15}{x} = 8 \\ \Longrightarrow\ & x^2 + 15 = 8x \\ \Longrightarrow\ & x^2 - 8x + 15 = 0. \end{aligned}$$
   (d) Factorising the above,
$$\begin{aligned} \Longrightarrow\ & (x-3)(x-5) = 0 \\ \Longrightarrow\ & x = 3, 5. \end{aligned}$$
   Hence, the side lengths are 3 cm and 5 cm.

2. We first multiply up by the denominator, and then gather like terms:
$$\begin{aligned} & \frac{2x-1}{2x+1} = 3 \\ \Longrightarrow\ & 2x - 1 = 3(2x+1) \\ \Longrightarrow\ & 2x - 1 = 6x + 3 \\ \Longrightarrow\ & -4 = 4x \\ \Longrightarrow\ & x = -1. \end{aligned}$$

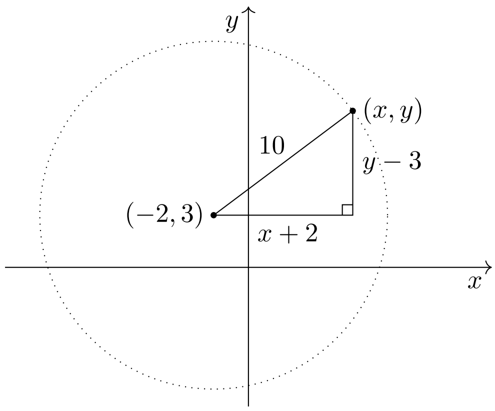
3. (a) By Pythagoras's theorem, the edge lengths are $\sqrt{1^2+1^2}$, $\sqrt{1^2+2^2}$ and $\sqrt{1^2+2^2}$. These add to $\sqrt{2} + 2\sqrt{5}$.
   (b) The dotted square has area 4, and the three unshaded right-angled triangles within it have areas $1, 1, \frac{1}{2}$. Therefore, the area of the shaded triangle is $4 - 1 - 1 - \frac{1}{2} = \frac{3}{2}$.

Nota Bene

The above technique is often the most efficient way of finding the area of a triangle (or other polygon) whose vertices are given as coordinates.

4. (a) Rewriting and grouping like terms,
$$\begin{aligned} & a + b \times c - d + d + c \times b - a \\ \equiv\ & a - a + bc + cb - d + d \\ \equiv\ & 2bc. \end{aligned}$$
   (b) Expanding and then factorising,
$$\begin{aligned} & ab(c-d) - bc(a-d) + ad(b-c) \\ \equiv\ & abc - abd - abc + bcd + abd - acd \\ \equiv\ & bcd - acd \\ \equiv\ & cd(b-a). \end{aligned}$$

5. (a) Completing the square for both x and y,
$$\begin{aligned} & x^2 + 4x + y^2 - 6y = 87 \\ \Longrightarrow\ & (x+2)^2 - 4 + (y-3)^2 - 9 = 87 \\ \Longrightarrow\ & (x+2)^2 + (y-3)^2 = 100. \end{aligned}$$
   (b) As shown in the diagram below, the centre is $(-2, 3)$ and the radius is 10.

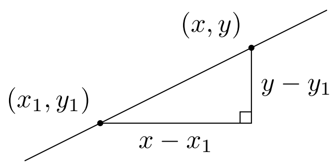


6. The first statement is false. A counterexample is $x = 1, y = 0$, for which $x^2y^2 = 0$ but $x^2 \neq 0$.

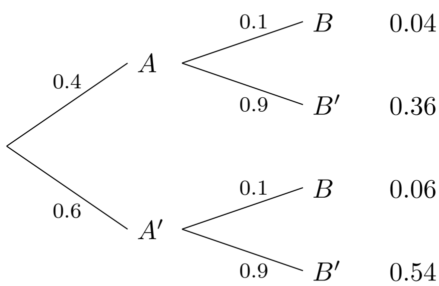
7. The gradient is $m = \frac{\Delta y}{\Delta x} = \frac{9}{3} = 3$. Substituting this and the point $(1, 2)$ into $y - y_1 = m(x - x_1)$, we get $y - 2 = 3(x-1)$, which is $y = 3x - 1$.

Nota Bene

The formula $y - y_1 = m(x - x_1)$, for a straight line, gradient m, through (x_1, y_1) is more efficient than substituting values into $y = mx + c$. Its proof is this diagram, in which the point (x_1, y_1) is fixed and the point (x, y) is variable:



8. Since A and B are independent, the second set of branches are identical, irrespective of whether A has occurred or not.



Summing the relevant branches gives
   (a) $\mathbb{P}(A \cap B) = 0.04$,
   (b) $\mathbb{P}(A \cup B) = 0.04 + 0.36 + 0.06 = 0.46$.

9. Since $(x-a) \equiv -(a-x)$,

$$\begin{aligned}&\sqrt{\frac{x-a}{x-b}} \times \sqrt{\frac{a-x}{b-x}}\\ \equiv\ &\sqrt{\frac{x-a}{x-b}} \times \sqrt{\frac{-(x-a)}{-(x-b)}}\\ \equiv\ &\sqrt{\frac{x-a}{x-b}} \times \sqrt{\frac{x-a}{x-b}}\\ \equiv\ &\frac{x-a}{x-b}.\end{aligned}$$

The condition $x > a, b$ ensures that $x-a$ and $x-b$ are both positive. This ensures no division by zero, and also that the square roots are well defined (positive inputs).

10. The "doubling time" means an increase of 1 on top of the original 1. At a rate of $\frac{1}{60}$ per month, this will take 60 months, or 5 years.

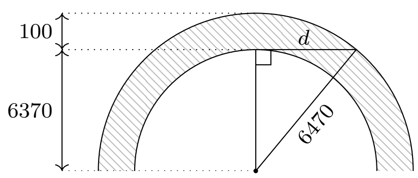
11. The equations are $a - b = 100$ and $ab = 5964$. We rearrange the former to give $a = 100 + b$, and substitute, yielding

$$\begin{aligned}&(100+b)b = 5964\\ \implies &b^2 + 100b - 5964 = 0\\ \implies &b = \frac{-100 \pm \sqrt{100^2 + 4 \cdot 5964}}{2}\\ &= 42, -142.\end{aligned}$$

So the integers are $(142, 42)$ or $(-42, -142)$.

12. (a) At the moment of sunset, light arrives from the horizon, i.e. tangentially to the ground. And the tangent to a sphere such as Earth, exactly like the tangent to a circle, is perpendicular to the radius.

(b) The radius of the atmosphere is approximately $6370 + 100 = 6470$, which, using (a), forms the hypotenuse of a right-angled triangle:



Hence, Pythagoras' theorem gives the distance travelled by the light d as

$$\begin{aligned}d &= \sqrt{6470^2 - 6370^2}\\ &= 1133\\ &\approx 11 \times 100.\end{aligned}$$

When the sun is overhead, sunlight passes through around 100 km of atmosphere. At sunset, it travels through around 11 times more.

13. In alphabetical order, the rearrangements are

AABB	ABAB	ABBA
BAAB	BABA	BBAA

Alphabetical listing proves that there are no other rearrangements.

14. (a) By the factor theorem, since $(x-4)$ is a factor, $x = 4$ is a root. Substituting $x = 4$, then, gives $24 + 4k = 0$, so $k = -6$.

(b) Solving with this k,

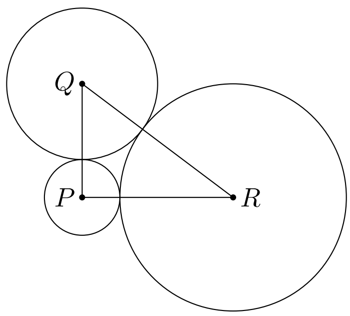
$$\begin{aligned}&x^2 - 6x + 8 = 0\\ \implies &(x-2)(x-4) = 0\\ \implies &x = 2, 4.\end{aligned}$$

15. (a) No. The lines are parallel and non-identical.

(b) Yes. The lines are not parallel.

16. A fraction can only equal zero if its numerator equals zero. This gives $x = 1, -3$. These would only fail to satisfy the equation if they were also roots of the denominator i.e. if they made the denominator zero. This is not the case. So, the solution is $x = 1, -3$.

17. Drawing in the radii, the scenario is



Summing radii, the distances $(|PQ|, |QR|, |RP|)$ are $(3, 4, 5)$. Since this is a Pythagorean triple with $3^2 + 4^2 = 5^2$, the triangle is right-angled.

18. Each factor simplifies by definition. The factor $p^{\log_p q}$ is literally "raise p by whatever power you need to raise p by in order to get q". You get q. So,

$$\begin{aligned}p^{\log_p q} \times q^{\log_q p} &\equiv q \times p\\ &\equiv pq.\end{aligned}$$

19. (a) By the factor theorem, since the y value of the curve is zero at -4 and -6, it has factors $(x+4)$ and $(x+6)$. Since it is a parabola, it can have no more algebraic factors than those two. It can, however, have a constant factor a.

(b) Substituting $x = 0, y = 48$, when get $24a = 48$, so $a = 2$.

20. The mean, IQR and standard deviation all have the original units. The variance, however, is a squared measure of spread: the formula contains the sum of squares $S_{xx} = \sum(x - \bar{x})^2$, with no square roots. So, its units are squared.

    (a) the mean has units ms^{-1},
    (b) the IQR has units ms^{-1},
    (c) the variance has units m^2s^{-2},
    (d) the standard deviation has units ms^{-1}.

21. The given lines are parallel, so the locus must be parallel to and halfway between them. Hence, it has equation $y = 4x + 8$.

22. We know that $p \propto {}^1\!/_{x^2}$ and $p \propto \sqrt{y}$. Hence, for some constants of proportionality a and b,

$$p = \frac{a}{x^2},$$
$$p = b\sqrt{y}.$$

Equating the two instances of p and squaring,

$$\frac{a}{x^2} = b\sqrt{y}$$
$$\implies \frac{a^2}{x^4} = b^2 y.$$

We can combine the constants of proportionality into a new constant $k = {}^{a^2}\!/_{b^2}$, to write

$$\frac{k}{x^4} = y.$$

Substituting $x = 1, y = 5$ yields $y = \dfrac{5}{x^4}$.

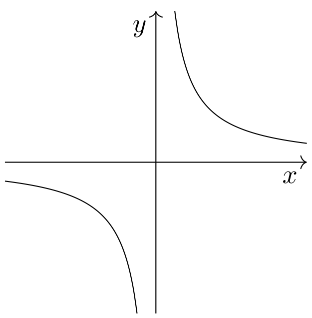
23. (a) Using the fact that $11 = 1.1 \times 10$,

$$\begin{aligned} & 5 \times 10^n + 6 \times 10^n \\ \equiv\ & 11 \times 10^n \\ \equiv\ & 1.1 \times 10^{n+1}, \end{aligned}$$

(b) Transferring a factor of 10,

$$\begin{aligned} & 5 \times 10^n \times 6 \times 10^n \\ \equiv\ & 30 \times 10^{2n} \\ \equiv\ & 3 \times 10^{2n+1}. \end{aligned}$$

24. The graph is the standard hyperbola:



25. Resultant force in the direction of the acceleration is $F - 30$, so the equation of motion is

$$F - 30 = 5 \times 4$$
$$\implies F = 20 + 30 = 50.$$

— Nota Bene —

No units are given in the solution, to match the fact that F is a number like 30. They are both numerical quantities **of** Newtons; they themselves do not represent quantities **in** Newtons. In other words, the force is 50 N, but F, in this question, is 50. By analogy, the expression *four fish* consists of the unitless number *four* and the unit *fish*.

26. The possibility space (set of all possible outcomes) is as follows, with the successful outcomes ticked:

	1	2	3	4	5	6
1						
2						✓
3					✓	
4				✓		
5			✓			
6		✓				

Therefore, since all 36 outcomes are equally likely, the probability is $\frac{5}{36}$.

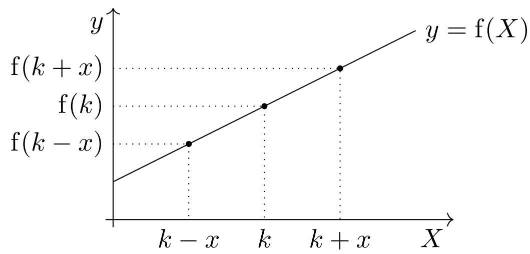
27. Simplifying the given function:

$$\begin{aligned} & \text{f}(k - x) + \text{f}(k + x) \\ \equiv\ & a(k - x) + b + a(k + x) + b \\ \equiv\ & ak - ax + ak + ax + 2b \\ \equiv\ & 2ak + 2b. \end{aligned}$$

This is independent of x.

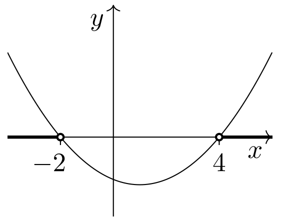
— Alternative Method —

Drawing a graph of $y = \text{f}(X)$, the values $k - x$ and $k + x$ are equidistant from k. Since f is linear, this means that $\text{f}(k - x)$ and $\text{f}(k + x)$ are equidistant from $\text{f}(k)$.



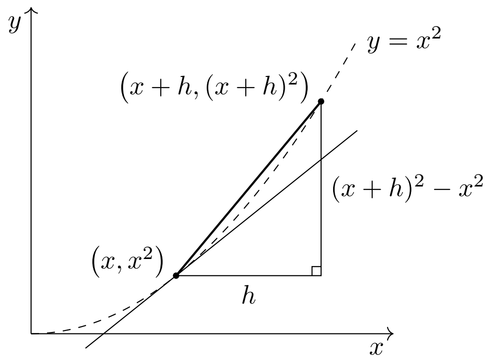
So, the sum of $\text{f}(k - x)$ and $\text{f}(k + x)$ is $2\,\text{f}(k)$, which is independent of x, as required.

28. (a) The boundary equation is $x^2 - 2x - 8 = 0$.
    (b) Solving the above, $x = 2, 4$.
    (c) $y = x^2 - 2x - 8$ is a positive parabola passing through $(-2, 0)$ and $(4, 0)$:



    (d) The inequality is satisfied by x values in the region outside of and not including the roots. In interval set notation, this can be expressed as $x \in (-\infty, -2) \cup (4, \infty)$.

29. (a) The fraction of which we are taking the limit is the gradient of a chord of the parabolic curve $y = x^2$, drawn from $\big(x, x^2\big)$, which we think of as fixed, to $\big(x + h, (x + h)^2\big)$, which we think of as mobile, depending on the value of h:



As $h \to 0$, the mobile right-hand point moves closer and closer to the fixed left-hand point, and the gradient of the chord gets closer and closer to the gradient of the tangent, which is the definition of "the gradient of the curve" at $\big(x, x^2\big)$. The *limit* is then the far boundary of this process: the tangent gradient towards which we are approaching.

    (b) Simplifying,

$$\begin{aligned} & (x+h)^2 - x^2 \\ \equiv\; & x^2 + 2xh + h^2 - x^2 \\ \equiv\; & 2xh + h^2 \\ \equiv\; & h(2x + h). \end{aligned}$$

    (c) The numerator has a common factor of h. So, we can divide top and bottom by h (which is not yet zero), and then safely take the limit:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &\equiv \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &\equiv \lim_{h\to 0} 2x + h \\ &\equiv 2x, \text{ as required.} \end{aligned}$$

30. Completing the square,

$$\begin{aligned} & 4x^2 + 24x + 54 \\ \equiv\; & 4(x^2 + 6x) + 54 \\ \equiv\; & 4(x+3)^2 - 4 \cdot 9 + 54 \\ \equiv\; & 4(x+3)^2 + 18. \end{aligned}$$

31. (a) The mapping f gives a well-defined function, because every input in D maps to precisely one output in C.
    (b) f is not invertible, for two reasons, each of which would be sufficient individually to stop f being one-to-one, hence invertible.
        i. Element #3 in C has no input mapping to it, so an inverse would have nothing to map that element back to,
        ii. Element #2 in C has two domain elements mapping to it (f is many-to-one). Hence, an inverse would be one-to-many, i.e. not a well-defined function.

32. Since the inequality is linear, we can solve directly, with a reversal of the direction of the inequality when dividing by -2:

$$\begin{aligned} & 3 - 2x > 4 \\ \iff\; & -2x > 1 \\ \iff\; & x < -\tfrac{1}{2} \\ \iff\; & x \in \big(-\infty, -\tfrac{1}{2}\big). \end{aligned}$$

——— Nota Bene ———

There are two equivalent ways to notate this set. Interval set notation $\big(-\infty, -\frac{1}{2}\big)$ is easiest. A more flexible set notation, which is clunkier here but has necessary uses elsewhere, is

$$\big\{x \in \mathbb{R} : x < -\tfrac{1}{2}\big\}.$$

$\mathbb{R}$ is the set of all real numbers, the $\in$ means "is an element of", and the colon means "such that".

33. The numbers are $x, 2x, 6x$, so $x + 2x + 6x = 18$. Solving this, $x = 2$. Therefore, the numbers are 2, 4 and 12.

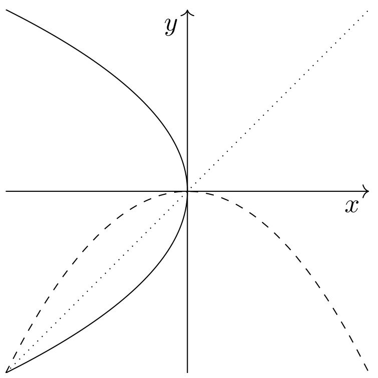
34. (a) Substituting $x = 2$ into the expression gives

$$5x^2 - 12x + 4\,\Big|_{x=2} = 0.$$

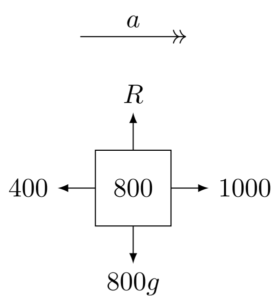
    (b) Since $x = 2$ is a root of $5x^2 - 12x + 4$, we know that $(x - 2)$ is a factor, by the factor theorem.

35. Substituting for b gives $a = (c + 1)^2$. Rearranging this gives $c = \pm\sqrt{a} - 1$.

36. The graph is a reflection of $y = -x^2$ (dashed) in the line $y = x$ (dotted):



37. (a) The force diagram is



(b) Horizontal $F = ma$ gives $1000 - 400 = 800a$, so $a = 0.75$ ms^{-2}.

(c) Substituting this acceleration into the formula $s = ut + \frac{1}{2}at^2$, the displacement is given by $s = 0 \cdot 20 + \frac{1}{2} \cdot 0.75 \cdot 20^2 = 150$ m.

38. (a) The possibility space is

HHH	HHT	HTH	HTT
THH	THT	TTH	TTT

(b) Since the eight outcomes are equally likely, counting outcomes in the possibility space gives the probability of two heads as $\frac{3}{8}$.

——— Alternative Method ———

Using the language of random variables, we have $X \sim B\big(3, \frac{1}{2}\big)$, and therefore

$$\begin{aligned}\mathbb{P}(X = 2) &= {}^3\mathrm{C}_2 \times \tfrac{1}{2}^1 \times \tfrac{1}{2}^2 \\ &= \tfrac{3}{8}.\end{aligned}$$

39. (a) Setting $y = 0$,

$$\begin{aligned} & x^2 - x - 6 = 0 \\ \Longrightarrow\ & (x-3)(x+2) = 0 \\ \Longrightarrow\ & x = -2, 3.\end{aligned}$$

(b) Using $y = x^n \implies \dfrac{dy}{dx} = nx^{n-1}$,

$$\begin{aligned} & y = x^2 - x - 6 \\ \Longrightarrow\ & \frac{dy}{dx} = 2x - 1.\end{aligned}$$

(c) Evaluating the gradient at the x intercepts,

$$\begin{aligned} & 2x - 1\big|_{x=-2,3} \\ &= 2 \times (-2) - 1 \text{ and } 2 \times 3 - 1 \\ &= \mp 5.\end{aligned}$$

(d) Every parabola has a line of symmetry. If there are two x intercepts, then this line lies at their midpoint. The x intercepts are images of one another, so the tangents at the x intercepts are also reflections of one another. Therefore, the gradients of the tangents must be $\pm m$.

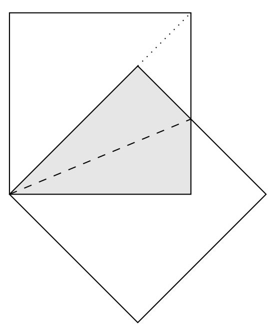
40. Rearranging and gathering terms in x,

$$\begin{aligned} & y = \frac{x}{x+a} \\ \Longrightarrow\ & (x+a)y = x \\ \Longrightarrow\ & xy + ay = x \\ \Longrightarrow\ & ay = x - xy \\ \Longrightarrow\ & ay = x(1-y) \\ \Longrightarrow\ & x = \frac{ay}{1-y}, \qquad y \neq 1.\end{aligned}$$

41. The boundary equation has been solved correctly. And each individual inequality comparing x with a number is correct, i.e. the original inequality is satisfied when $1/3 < x$ or when $x < -1/3$. But these can't happen together. So, the grouping into one simultaneous inequality is incorrect notation. By proxy, it claims that $1/3 < -1/3$, which isn't true.

Such an inequality, consisting of two *alternative* regions, should be written as two alternatives, i.e. "$x < -1/3$ or $x > 1/3$". Interval set notation is even clearer: $x \in (-\infty, -1/3) \cup (1/3, \infty)$.

42. The diagonals of the squares have length $\sqrt{2}$, by Pythagoras. These are split into 1 (the side length of the other square) and what remains (this is shown dotted below).



The dotted line, therefore, has length $\sqrt{2} - 1$. By symmetry, the short sides of the shaded kite also have length $\sqrt{2} - 1$. The area of the kite is then that of two triangles:

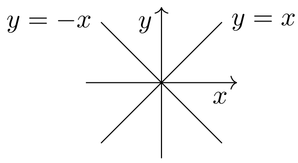
$$\begin{aligned} A &= 2 \times \tfrac{1}{2} \times 1 \times (\sqrt{2} - 1) \\ &= \sqrt{2} - 1, \text{ as required.}\end{aligned}$$

43. Taking logs base 3 of both sides,

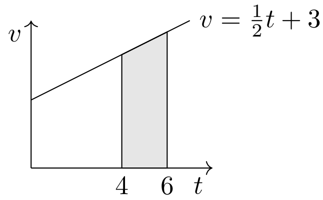
$$\begin{aligned} 3^{x+1} &= 7 \\ \implies x+1 &= \log_3 7 \\ \implies x &= \log_3 7 - 1 \\ &= 0.771 \quad \text{(3sf)}. \end{aligned}$$

44. In the equation of a circle, the x^2 and y^2 terms must have the same coefficient, so $a = 1$. Then, since the centre is on $y = x$, $b = 4$. (You could complete the square to verify this.) And, because $(0, 0)$ must satisfy the equation, we get $c = 0$.

45. Square rooting, we get $x = \pm y$, which offers two options, $y = x$ and $y = -x$. The graph is therefore a pair of intersecting lines:



46. (a) The velocity-time graph is:



(b)
$$\begin{aligned} \Delta x &= \tfrac{1}{2}(a+b)h \\ &= \tfrac{1}{2}\left[\left(\tfrac{4}{2}+3\right) + \left(\tfrac{6}{2}+3\right)\right] \cdot 2 \\ &= 11. \end{aligned}$$

47. By Pythagoras, the square's diagonals have length $\sqrt{2}$, so the line consists of four unit segments, and six segments of length $\sqrt{2}/2$. Hence, the total length is $4 + 6 \cdot \sqrt{2}/2 = 4 + 3\sqrt{2}$.

48. This is already factorised, so we can solve directly:

$$\begin{aligned} &(x^2+1)^7(3x-2) = 0 \\ \implies &(x^2+1)^7 = 0 \text{ or } (3x-2) = 0. \end{aligned}$$

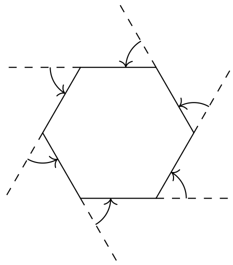
But the first equation has no roots, since $(x^2 + 1)$ is always positive. Hence, $3x - 2 = 0$, so $x = \frac{2}{3}$.

49. Maximum or minimum acceleration occurs if the forces are parallel or antiparallel, which gives the resultant force as either $5m$ or m Newtons. Hence, $a_{\text{max}} = 5 \text{ ms}^{-2}$ and $a_{\text{min}} = 1 \text{ ms}^{-2}$.

— Nota Bene —

Unlike "parallel", which can be used to describe both lines and vectors, "antiparallel" can only be used to describe vectors. This is because, unlike vectors, lines have no sense of $\pm$ direction.

50. One circuit of the perimeter of the n-gon, via n vertices, involves turning 2π radians (360°).



Hence, each exterior angle (the angles turned through at each vertex, marked with arrows above) is $\frac{2\pi}{n}$ radians.

51. Completing the square for x and y gives

$$\begin{aligned} &x^2 + 6x + y^2 - 3y = 0 \\ \implies &(x+3)^2 - 9 + (y - \tfrac{3}{2})^2 - \tfrac{9}{4} = 0 \\ \implies &(x+3)^2 + (y - \tfrac{3}{2})^2 = \tfrac{45}{4}. \end{aligned}$$

The equation $(x - a)^2 + (y - b)^2 = r^2$ is that of a circle centred on (a, b), with radius r. So, $x^2 + 6x + y^2 - 3y = 0$ is a circle, centre $(-3, 3/2)$ and radius $\sqrt{45}/2$.

52. Writing the sums out longhand,

(a) $\underbrace{1 + 1 + ... + 1}_{\text{10 times}} = 10$,

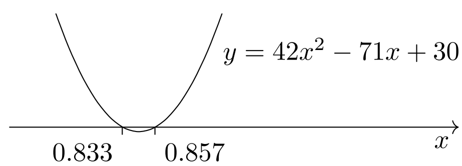
(b) $\underbrace{1 + 1 + ... + 1}_{n+1 \text{ times}} = n + 1$,

(c) $\underbrace{a + a + ... + a}_{n \text{ times}} = an$.

53. Using the quadratic formula, the roots are

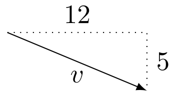
$$\begin{aligned} x &= \frac{71 \pm \sqrt{71^2 - 4 \cdot 42 \cdot 30}}{2 \cdot 42} \\ &= \frac{70}{84} \text{ or } \frac{72}{84} \\ &= 0.833... \text{ or } 0.857... \end{aligned}$$

These roots both lie between 0.8 and 0.9, so, unless a decimal search starts at at least 2dp accuracy, it will fail to spot a sign change: all values reported by the algorithm will be positive.



The algorithm is not sensitive enough to detect roots which are close together.

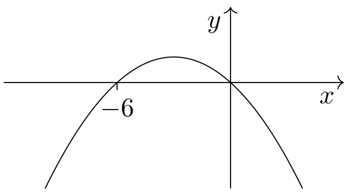
54. Vectors **i** and **j** are perpendicular unit vectors:



The speed is the magnitude v of the velocity **v**. Pythagoras gives $v = \sqrt{12^2 + 5^2} = 13 \text{ ms}^{-1}$.

55. (a) The boundary equation is $-x^2 - 6x = 0$, which has roots $x = 0, -6$.

(b) $y = -x^2 - 6x$ is a negative parabola with single roots at $x = -6$ and $x = 0$:



(c) We want x such that the graph is above or on the x axis. The full solution is

$$\begin{aligned} -x(x+6) &\geq 0 \\ \Longrightarrow x &\in [-6, 0]. \end{aligned}$$

——— Nota Bene ———

The following notations, both of which are types of set notation, are identical.

Interval set notation	Set notation
$(0,1)$	$\{x \in \mathbb{R} : 0 < x < 1\}$
$(0,1]$	$\{x \in \mathbb{R} : 0 < x \leq 1\}$
$[0,1)$	$\{x \in \mathbb{R} : 0 \leq x < 1\}$
$[0,1]$	$\{x \in \mathbb{R} : 0 \leq x \leq 1\}$

Generally, interval notation is better. But the less succinct notation comes into its own when discussing e.g. rationals. There is no interval set notation for $\{x \in \mathbb{Q} : 0 \leq x \leq 1\}$, which is rational numbers between 0 and 1.

56. The quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $\Delta > 0$, then the formula gives two distinct roots. If $\Delta = 0$, the square root is zero, so the formula gives one root. If $\Delta < 0$, there is no real value of $\sqrt{\Delta}$, so the equation has no real roots.

57. The equation of motion is

$$\begin{aligned} 34 - 10 &= 3m \\ \Longrightarrow m &= \tfrac{24}{3} = 8 \text{ kg}. \end{aligned}$$

——— Nota Bene ———

The "equation of motion" is another way of saying Newton's second law. This is also notated NII in these books and N2 elsewhere.

58. Multiplying out and differentiating term by term:

$$\begin{aligned} \mathrm{f}(x) &= 4x(1 - x^2) = 4x - 4x^3 \\ \Longrightarrow \mathrm{f}'(x) &= 4 - 12x^2. \end{aligned}$$

——— Alternative Method ———

Using the product rule,

$$\begin{aligned} \mathrm{f}(x) &= 4x(1 - x^2) \\ \Longrightarrow \mathrm{f}'(x) &= (4x)'(1 - x^2) + 4x(1 - x^2)' \\ &\equiv 4(1 - x^2) + 4x(-2x) \\ &\equiv 4 - 12x^2. \end{aligned}$$

59. Quoting standard results:

(a) $3! = 6$,

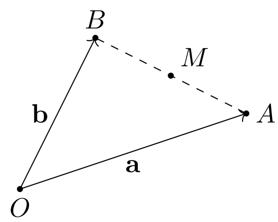
(b) ${}^3\mathrm{C}_1 = 3$.

60. There are π radians in a semicircle. $1\frac{1}{2}$ right angles is $\frac{3}{4}$ of a semicircle, giving $\frac{3\pi}{4}$ radians.

——— Nota Bene ———

When an angle in radians is a simple multiple of π, e.g. $\frac{3\pi}{4}$ but not 4.0373π, it's usual to leave the angle in terms of π. The answer 2.36 radians (3sf) is correct, but it's less useful than $\frac{3\pi}{4}$ radians.

61. Sketching, with A and B in arbitrary locations:



(a) The journey $\overrightarrow{AB}$ is the same as: back from A to O then forwards from O to B. Therefore,

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &\equiv -\overrightarrow{OA} + \overrightarrow{OB} \\ &= \mathbf{b} - \mathbf{a}. \end{aligned}$$

(b) The position vector of a midpoint is the mean of the position vectors: $\overrightarrow{OM} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.

62. By the factor theorem, the roots of the equation are $p, q, \frac{1}{2}q$. These must be $2, 3, 4$ in some order. There is only one order that works. We require $\frac{1}{2}q = 2$, $p = 3$ and $q = 4$.

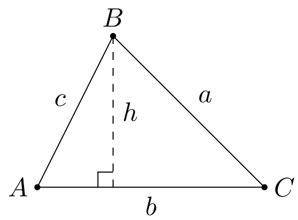
63. The day of the week on which a birthday falls is not independent in any two given years. In fact, a birthday can't fall on the same day in consecutive years because a year is 52 weeks and 1 day (or 2 in a leap year). So, the relevant probability is zero.

64. This biquadratic $\left(\text{quadratic in } x^2\right)$ equation can be factorised as follows:

$$\begin{aligned} & x^4 - x^2 - 72 = 0 \\ \Longrightarrow\ & (x^2-9)(x^2+8) = 0 \\ \Longrightarrow\ & x^2 = 9 \text{ or } -8. \end{aligned}$$

The latter has no real roots, so $x = \pm 3$.

65. $\triangle ABC$, with a perpendicular dropped from vertex B to side AC, is as follows:



Using the triangle on the right, the perpendicular height h is given by $h = a \sin C$. Hence,

$$\begin{aligned} A_\triangle &= \tfrac{1}{2}\text{base} \times \text{height} \\ &= \tfrac{1}{2} b \times a \sin C \\ &\equiv \tfrac{1}{2} ab \sin C, \text{ as required.} \end{aligned}$$

66. (a) $\mathrm{gf}(2) = \mathrm{g}(3) = 4$,
(b) $\mathrm{f}^{-1}\mathrm{g}^{-1}(4) = \mathrm{f}^{-1}(3) = 2$,
(c) $\mathrm{g}^{-1}\mathrm{f}^2(1) = \mathrm{g}^{-1}\mathrm{f}(2) = \mathrm{g}^{-1}(3) = 2$.

Nota Bene

The notation $\mathrm{f}^2(x)$ denotes the application of the function f twice to the input x:

$$\mathrm{f}^2(x) \equiv \mathrm{f}\big(\mathrm{f}(x)\big).$$

But this notation *doesn't* carry over to e.g. $\sin^2 x$, which, exceptionally, means $(\sin x)^2$. The squared trig functions are so common (due to Pythagoras's theorem) that they have their own notation. If it wasn't for the notation $\sin^2 x \equiv \sin\big(\sin(x)\big)$ and so on, we would get inundated with brackets.

67. By definition of a logarithm, $\log_a a^2 + \log_b b^3$ reads: "the power you need to raise a by to get a^2, added to the power you need to raise b by to get b^3. This is $2 + 3 = 5$.

68. Starting with the RHS,

$$\begin{aligned} & \frac{1}{p(p+q)} + \frac{1}{q(p+q)} \\ \equiv\ & \frac{q}{pq(p+q)} + \frac{p}{pq(p+q)} \\ \equiv\ & \frac{p+q}{pq(p+q)} \\ \equiv\ & \frac{1}{pq}, \text{ as required.} \end{aligned}$$

69. Rewriting the modulus algebra,

$$\begin{aligned} & |2x-1| = 9 \\ \Longrightarrow\ & 2x - 1 = 9, -9 \\ \Longrightarrow\ & x = 5, -4. \end{aligned}$$

Nota Bene

Often, when solving modulus algebra, you need to take a graphical approach. In some cases, however, direct implication is possible. This is one-way:

$$\begin{aligned} & |a| = b \\ \Longrightarrow\ & a = \pm b. \end{aligned}$$

The following implications are two-way, giving three ways of expressing exactly the same fact:

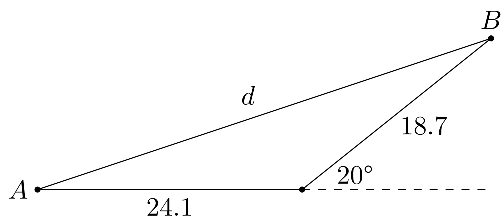
$$\begin{aligned} & |a| = |b| \\ \Longleftrightarrow\ & a^2 = b^2 \\ \Longleftrightarrow\ & a = \pm b. \end{aligned}$$

70. Division by zero occurs at $x = -c$ and $x = d$, so these are the vertical asymptotes of the curve.

Nota Bene

If e.g. $a = c$ and $b = d$, then the curve would have been undefined at $x = c$ and $x = -d$ (division by zero), but would not have generated an asymptote, since its value would have been 1 wherever defined. However, since we are told that $\pm a, \pm b, \pm c, \pm d$ are distinct real numbers, this possibility is ruled out.

71. Approximately to scale, the scenario is



Using the cosine rule:

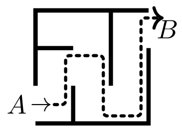
$$\begin{aligned} d^2 &= 24.1^2 + 18.7^2 - 2 \cdot 24.1 \cdot 18.7 \cos 160° \\ &= 1777.4825... \end{aligned}$$

Hence, $d = 42.16$. This gives the extra distance as

$$\begin{aligned} d_{\text{extra}} &= 24.1 + 18.7 - 42.16 \\ &= 0.63979... \\ &= 0.640 \text{ nautical miles (3sf).} \end{aligned}$$

72. False, because $x = 1$ is parallel to the y axis. No polynomial curve $y = \mathrm{f}(x)$ is ever parallel to the y axis: the gradient formula $\mathrm{f}'(x)$ of a polynomial is always finite.

73. The shortest route through the maze has 6 turns:



The probability of choosing the right direction at every turn is given by $\frac{1}{2}^6 = \frac{1}{64}$.

NOTA BENE

You might have interpreted the Q as offering the possibility that, having turned left for the first time, the robot could slip by the next short wall to hit the uppermost boundary of the square. In this case, the solution is unchanged.

74. (a) Since the range (set of attained outputs) has a minimum but not a maximum, the parabola must be positive.

(b) For the vertex (stationary point) $f'(x) = 0$, which gives $2ax + 4 = 0$, so $x = -\frac{2}{a}$.

ALTERNATIVE METHOD

Completing the square,

$$\begin{aligned} & ax^2 + 4x + 15 \\ \equiv\ & a\left(x^2 + \tfrac{4}{a}x\right) + 15 \\ \equiv\ & a\left(\left(x + \tfrac{2}{a}\right)^2 - \tfrac{4}{a^2}\right) + 15. \end{aligned}$$

So, the minimum value is at $x = -\frac{2}{a}$.

(c) Substituting $x = -\frac{2}{a}$ into $f(x)$ and equating to 7 gives

$$\begin{aligned} & a\left(-\tfrac{2}{a}\right)^2 + 4\left(-\tfrac{2}{a}\right) + 15 = 7 \\ \Longrightarrow\ & \tfrac{4}{a} - \tfrac{8}{a} = -8 \\ \Longrightarrow\ & \tfrac{4}{a} = 8 \\ \Longrightarrow\ & a = \tfrac{1}{2}. \end{aligned}$$

75. (a) The equation $x = 4$ has no derivative with respect to x, as x cannot vary. On an (x, y) graph, the equation defines a vertical line. Any attempted calculation of the gradient of such as line involves division by zero. An algebraic attempt to differentiate both sides gives

$$\begin{aligned} & \tfrac{d}{dx}(x) = \tfrac{d}{dx}(4) \\ \Longrightarrow\ & 1 = 0. \end{aligned}$$

The algebraic contradiction correctly reflects the impossibility of the operation.

(b) The equation $y = 4$ describes a horizontal line, so its derivative is well defined. The gradient is everywhere zero: $\frac{dy}{dx} = 0$.

76. By the factor theorem,

$$\begin{aligned} & (3x - 7)(x^2 + 1)(x^2 - 4) = 0 \\ \Longrightarrow\ & (3x - 7) = 0 \text{ or } (x^2 + 1) = 0 \text{ or } (x^2 - 4) = 0. \end{aligned}$$

These have, in order, a root at $x = 7/3$, no real roots (since $x^2 \geq 0$) and roots at $x = \pm 2$. So, the solution set is $x \in \{-2, 2, 7/3\}$.

77. The notation $\left[F(x)\right]_a^b$ means $F(b) - F(a)$. So,

$$\begin{aligned} & \left[x^2 + x + 1\right]_0^4 \\ =\ & (16 + 4 + 1) - (0 + 0 + 1) \\ =\ & 20. \end{aligned}$$

78. The mean is as a measure of central tendency. As such, it is affected by both a (scaling the data) and b (translating the data). This gives the mean of the coded sample as $\bar{y} = a\bar{x} + b$.

The variance is a measure of spread. As such, it is unaffected by the translation $+b$. It is affected by the scaling, however. As a squared measure, it is scaled accordingly: $s_y^2 = a^2 s_x^2$.

79. In each case, we find an element x of the first set A that is not an element of the second set B. That way $x \in A$, so the first statement holds, but $x \notin B$, so the second statement doesn't hold. In each case, this disproves the implication $x \in A \implies x \in B$.

(a) Any negative integer, e.g. -1.

(b) Any fraction, e.g. $\frac{1}{2}$.

(c) Any irrational number, e.g. $\sqrt{2}$.

80. We multiply numerator and denominator by the conjugate of the denominator, thereby producing a difference of two squares:

$$\begin{aligned} & \frac{3}{\sqrt{7} - 2} \\ =\ & \frac{3(\sqrt{7} + 2)}{(\sqrt{7} - 2)(\sqrt{7} + 2)} \\ =\ & \frac{3\sqrt{7} + 6}{7 - 4} \\ =\ & \sqrt{7} + 2. \end{aligned}$$

81. Pythagoras on $\triangle ABC$ tells us that $|AC| = \sqrt{2}$. Hence, $\triangle AXC$ has sides of length $(1, 1, \sqrt{2})$. So, $\triangle AXC$ is congruent to $\triangle ABC$ (condition SSS), and is therefore right-angled, as required.

82. Any recurring decimal, such as $\frac{1}{9} = 0.1111...$, is a counterexample.

83. Substituting into the LHS of the identity,

$$\begin{aligned}&(a\,\mathrm{f}(x))^2 + (b\,\mathrm{f}'(x))^2\\ \equiv\ &(a\sin\tfrac{a}{b}x)^2 + (b\tfrac{a}{b}\cos\tfrac{a}{b}x)^2\\ \equiv\ &a^2\sin^2\tfrac{a}{b}x + a^2\cos^2\tfrac{a}{b}x\\ \equiv\ &a^2(\sin^2\tfrac{a}{b}x + \cos^2\tfrac{a}{b}x).\end{aligned}$$

By the first Pythagorean trig identity, which is $\sin^2\theta + \cos^2\theta \equiv 1$, this is a^2, as required.

— NOTA BENE —

The three Pythagorean trig identities are

$$\begin{aligned}&①: \sin^2\theta + \cos^2\theta \equiv 1\\ &②: \tan^2\theta + 1 \equiv \sec^2\theta\\ &③: 1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta.\end{aligned}$$

① is true by definition of sin and cos (unit circle). ② is proved by dividing ① by $\cos^2\theta$ and using

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}.$$

③ is proved by dividing ① by $\sin^2\theta$ and using

$$\cot\theta \equiv \frac{\cos\theta}{\sin\theta}.$$

84. There are $2^3 = 8$ equally likely outcomes, of which HHH and TTT are successful. So, $p = 2/8 = 1/4$.

85. The average rate of change of y with respect to x between the first and second columns is

$$\frac{\Delta y}{\Delta x} = \frac{4-0}{3-1} = 2.$$

Then, between the second and third columns, it is

$$\frac{\Delta y}{\Delta x} = \frac{9-4}{6-3} = \frac{5}{3}.$$

In a linear relationship, these rates would be the same. So, x and y cannot be related linearly.

86. By the factor theorem, $2^x = -1, 8$. The former has no roots, as $2^x > 0$. So, the solution is $x = 3$.

87. A runs the race in $\frac{1000}{5} = 200$ seconds, while B runs it in $\frac{1000}{4} = 250$ seconds. Hence, if B starts 50 seconds before A, they will finish together.

88. Using the fact that $x - p \equiv -(p - x)$,

$$\begin{aligned}&\sqrt{\frac{x-p}{x-q}} \div \sqrt{\frac{q-x}{p-x}}\\ \equiv\ &\sqrt{\frac{x-p}{x-q}} \times \sqrt{\frac{p-x}{q-x}}\\ \equiv\ &\sqrt{\frac{x-p}{x-q}} \times \sqrt{\frac{x-p}{x-q}}\\ \equiv\ &\frac{x-p}{x-q}.\end{aligned}$$

89. The squared distances to $(0, 1)$ are $2^2 + 3^2 = 13$ and $4^2 + 1^2 = 17$. So $(3, 3)$ is closer.

— NOTA BENE —

It is often easier to work with squared distances, particularly in comparison, rather than with the distances themselves.

90. By the definition of a logarithm, $\log_a b$ is "what you have to raise a by to get b". This gives

(a) $\log_a \sqrt[3]{a} = \log_a a^{\frac{1}{3}} = \frac{1}{3}$.

(b) $\log_{a^2} a = \frac{1}{2}$.

(c) $\log_{a^3} a^2 = \frac{2}{3}$.

— ALTERNATIVE METHOD —

In (b) and (c), you can also use the fact that, if you raise base and input to the same power, the value of a log is unchanged. (This is analogous to multiplying top and bottom of a fraction by the same number.) Hence, $\log_{a^2} a = \log_a a^{\frac{1}{2}} = 1/2$ and $\log_{a^3} a^2 = \log_a a^{\frac{2}{3}} = 2/3$.

91. The straight line is $y = -\frac{2}{3}x + \frac{5}{3}$, with gradient $m = -\frac{2}{3}$. Hence, $\frac{dy}{dx} = -\frac{2}{3}$.

— ALTERNATIVE METHOD —

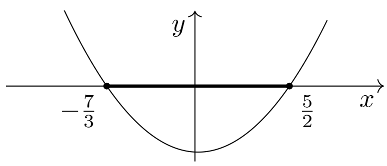
Differentiating both sides with respect to x,

$$\begin{aligned}&2x + 3y = 5\\ \Longrightarrow\ &2 + 3\tfrac{dy}{dx} = 0\\ \Longrightarrow\ &\tfrac{dy}{dx} = -\tfrac{2}{3}.\end{aligned}$$

92. (a) Setting $\mathrm{g}(x) = 0$ and factorising,

$$\begin{aligned}&6x^2 - x - 35 = 0\\ \Longrightarrow\ &(3x+7)(2x-5) = 0\\ \Longrightarrow\ &x = -\tfrac{7}{3}, \tfrac{5}{2}.\end{aligned}$$

(b) Sketch of $y = 6x^2 - x - 35$, with solution set for part (c) marked:



(c) The solution set, for which the curve is on or below the x axis, is between and including the roots $[-7/3, 5/2]$.

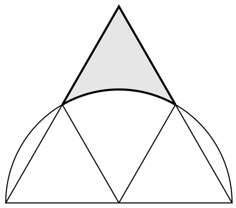
— ALTERNATIVE METHOD —

The solution set could also be expressed as $\{x \in \mathbb{R} : -7/3 \le x \le 5/2\}$.

93. There are 2π radians in 360°. Applying this as a scale factor,

$$2.6 \times \frac{360°}{2\pi} = 149° \text{ (0dp)}.$$

94. The radii to the points of intersection produce equilateral triangles of side length $\frac{1}{2}l$:



The arc of the arrowhead subtends an angle of 60° at the centre, with radius $\frac{1}{2}l$. So, its length is

$$\begin{aligned} s &= \tfrac{1}{6} \times 2\pi \cdot \tfrac{1}{2}l \\ &\equiv \tfrac{1}{3}\pi l. \end{aligned}$$

The straight sides of the arrowhead have length $\frac{1}{2}l$, giving

$$\begin{aligned} P &= \tfrac{1}{2}l + \tfrac{1}{2}l + \tfrac{1}{3}\pi l \\ &\equiv l\big(1 + \tfrac{1}{3}\pi\big). \end{aligned}$$

95. The equation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} t.$$

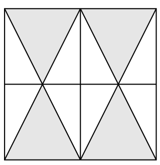
At $t = 0$, the curve is at $(3, -1)$. The gradient of the curve is $\frac{5}{2}$, as $\frac{dy}{dt} = 5$ and $\frac{dx}{dt} = 2$. Substituting into $y - y_1 = m(x - x_1)$,

$$\begin{aligned} y + 1 &= \tfrac{5}{2}(x - 3) \\ \Longrightarrow -5x + 2y &= 17. \end{aligned}$$

——— Nota Bene ———

If you think of the variable t as representing time, then the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are *velocities* in the x and y directions.

96. Adding two lines:

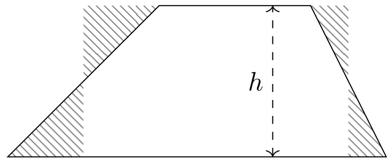


Counting congruent triangles, we can see that half of the total area is shaded.

97. For each set of data, $\sum x$ is given by $n_i\bar{x}_i$. So, the combined mean is the sum of these values, divided by the total number of data $n_1 + n_2$. This is

$$\bar{x}_{\text{total}} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}.$$

98. A trapezium can be dissected at the midpoints of its non-parallel sides and rearranged as follows:



Since the dissection is at the midpoints, the length of the rectangle formed is the average of a and b. Hence, $A = \frac{1}{2}(a + b)h$.

99. (a) Any a, b with $a < b < 0$. For example,

$$-3 < -2, \text{ but } 9 \not< 4.$$

(b) Any a, b with $0 < a < b < 1$. For example,

$$\tfrac{1}{3} < \tfrac{1}{2}, \text{ but } 3 \not< 2.$$

100. (a) $6 + 8 = 10a$ gives $a = 1.4$ ms^{-2}.

(b) The forces are perpendicular, so the resultant is given by Pythagoras. Since $6^2 + 8^2 = 10^2$ the acceleration is $a = 1$ ms^{-2}.

(c) $8 - 6 = 10a$ gives $a = 0.2$ ms^{-2}.

——— End of 1st Hundred ———